

SPORADIC SIMPLE GROUPS AND THE *FSZ* PROPERTIES

MARC KEILBERG

ABSTRACT. We investigate a possible connection between the *FSZ* properties of a group and its Sylow subgroups. We show that the simple group $G_2(5)$ and all sporadic simple groups with order divisible by 5^6 are not *FSZ*, and that neither are their Sylow 5-subgroups. The groups $G_2(5)$ and HN were previously established as non-*FSZ* by Peter Schauenburg; we present alternative proofs. All other sporadic simple groups and their Sylow subgroups are shown to be *FSZ*. We also consider all perfect groups available through GAP with order at most 10^6 , and show they are non-*FSZ* if and only if their Sylow 5-subgroups are non-*FSZ*.

INTRODUCTION

The *FSZ* properties for groups, as introduced by Iovanov et al. [3], arise from considerations of certain invariants of the representation categories of semisimple Hopf algebras [4, 9, 10]. See [8] for a detailed discussion of the wide ranging uses and generalizations of these invariants. When applied to Drinfeld doubles of finite groups, these invariants are described entirely in group theoretical terms, and are in particular invariants of the group itself. The *FSZ* property is then concerned with whether or not these invariants are always integers—which gives the Z in *FSZ*. However, the properties seem largely immune to the usual group theoretical techniques, which has thus far made both theoretical and computational explorations of the properties difficult.

While the *FSZ* and non-*FSZ* group properties are well-behaved with respect to direct products, there is currently little reason to suspect a particularly strong connection to proper subgroups which are not direct factors. Indeed, by [3] the symmetric groups S_n are *FSZ*, while there exists non-*FSZ* groups of order 5^6 . Therefore, S_n is *FSZ* but contains non-*FSZ* subgroups for all sufficiently large n . On the other hand, non-*FSZ* groups can have every proper subquotient be *FSZ*. Even the known connection to the one element centralizers—see the comment following Definition 1.1—is weak. In this paper we will establish a few simple improvements to this situation, and then proceed to establish a number of examples of *FSZ* and non-*FSZ* groups that suggest a potential non-trivial connection to Sylow subgroups.

We will make extensive use of GAP [2] and the AtlasRep[13] package. Most of the calculations were designed to be calculated with only 2GB of memory or

2010 *Mathematics Subject Classification.* Primary: 20D08; Secondary: 20F99, 16T05, 18D10.

Key words and phrases. sporadic simple groups, Monster group, Baby Monster group, Harada-Norton group, Lyons group, higher Frobenius-Schur indicators, FSZ groups, Sylow subgroups.

This work is in part an outgrowth of an extended e-mail discussion between Geoff Mason, Susan Montgomery, Peter Schauenburg, Miodrag Iovanov, and the author. The author thanks everyone involved for their contributions, feedback, and encouragement.

less available—in particular, using only a 32-bit implementation of GAP—, though in a few cases a larger workspace was necessary. In all cases the calculations can be completed in workspaces with no more than 10GB of memory available. The author ran the code on an Intel(R) Core(TM) i7-4770 CPU @ 3.40GHz machine with 12GB of memory. Most of the calculations were completed in a matter of minutes or less, while the longest calculation takes approximately two days.

The structure of the paper is as follows. We introduce the relevant notation, definitions, and background information in Section 1. In Section 2 we present a few simple results which offer some connections between the FSZ (or non- FSZ) property of G and certain of its subgroups. The remainder of the paper will be dedicated to exhibiting a number of examples that support the idea of a stronger connection to Sylow subgroups than occurs in Corollary 2.5. In Section 3 we show that the simple groups $G_2(5)$, HN , Ly , B , and M , as well as their Sylow 5-subgroups, are all non- FSZ_5 . In Section 4 we show that all other sporadic simple groups (including the Tits group) and their Sylow subgroups are FSZ ; this is summarized in Theorem 4.4. We finish our examples in Section 5 by examining those perfect groups available through GAP [2], and show that they are FSZ if and only if their Sylow subgroups are FSZ —indeed, they are non- FSZ if and only if their Sylow 5-subgroup is non- FSZ_5 . Of necessity, these results also establish that various centralizers and maximal subgroups in the groups in question are also non- FSZ_5 , which can be taken as additional examples. If the reader is interested in FSZ properties for other simple groups, we note that Schauenburg [12] has checked all simple groups of order at most 200 million, and that several families of simple groups were established as FSZ by Iovanov et al. [3]. In Appendix A we define a few helpful GAP [2] functions.

While Theorem 4.4 shows a curious connection between sporadic simple groups and the (non-) FSZ_5 property, the constant recurrence of the number 5 and Sylow 5-subgroups of order 5^6 in the subsequent is otherwise more of a computationally convenient coincidence. The reasons for this will be mentioned during the course of the paper.

1. BACKGROUND AND NOTATION

The study of FSZ groups is connected to the following sets.

Definition 1.1. Let G be a group, $u, g \in G$, and $m \in \mathbb{N}$. Then we define

$$G_m(u, g) = \{a \in G : a^m = (au^{-1})^m = g\}.$$

Note that $G_m(u, g) = \emptyset$ if $u \notin C_G(g)$, and that in all cases $G_m(u, g) \subseteq C_G(g)$. In particular, letting $H = C_G(g)$, then when $u \in H$ we have

$$G_m(u, g) = H_m(u, g).$$

The following will then serve as our definition of the FSZ_m property. It's equivalence to other definitions follows easily from [3, Corollary 3.2] and applications of the Chinese remainder theorem.

Definition 1.2. A group G is FSZ_m if and only if for all $g \in G$, $u \in C_G(g)$, and $n \in \mathbb{N}$ coprime to the order of g , we have

$$G_m(u, g) = G_m(u, g^n).$$

We say a group is FSZ if it is FSZ_m for all m .

All expressions of the form $G_m(u, g^n)$ will implicitly assume that n is coprime to the order of g . We are free to replace n with an equivalent value which is coprime to $|G|$ whenever necessary. Moreover, when computing cardinalities $|G_m(u, g)|$ it suffices to compute the cardinalities $|H_m(u, g)|$ for $H = C_G(g)$, instead. This latter fact is very useful when attempting to work with groups of large order, or groups with centralizers that are easy to compute in, especially when the group is suspected of being non- FSZ .

Remark 1.3. There are stronger conditions called FSZ_m^+ , the union of which yields the FSZ^+ condition, which are also introduced by Iovanov et al. [3]. The FSZ_m^+ condition is equivalent to the centralizer of every non-identity element with order not in $\{1, 2, 3, 4, 6\}$ being FSZ_m , which is in turn equivalent to the sets $G_m(u, g)$ and $G_m(u, g^n)$ being isomorphic permutation modules for the two element centralizer $C_G(u, g)$ [3, Theorem 3.8], with u, g, n satisfying the same constraints as for the FSZ_m property. Here the action is by conjugation. It is not known if this condition is strictly stronger than the FSZ_m condition, and we do not consider the matter here, as our focus is mostly on non- FSZ groups. We note that while the FSZ property is concerned with certain invariants being in \mathbb{Z} , the FSZ^+ property is not concerned with these invariants being non-negative integers. When the invariants are guaranteed to be non-negative is another area of research, and will also not be considered here.

Example 1.4. The author has shown that quaternion groups and certain semidirect products defined from cyclic groups are always FSZ [6, 7]. This includes the dihedral groups, semidihedral groups, and quasidihedral groups, among many others.

Example 1.5. Iovanov et al. [3] showed that several groups and families of groups are FSZ , including:

- All regular p -groups.
- $\mathbb{Z}_p \wr \mathbb{Z}_p$, the Sylow p -subgroup of S_{p^2} , which is an irregular FSZ p -group.
- $PSL_2(q)$ for a prime power q .
- Any direct product of FSZ groups. Indeed, any direct product of FSZ_m groups is also FSZ_m , as the cardinalities of the sets in Definition 1.1 split over the direct product in an obvious fashion.
- The Mathieu groups M_{11} and M_{12} .
- Symmetric and alternating groups.

Because of the first item, Susan Montgomery has proposed that we use the term FS -regular instead of FSZ , and FS -irregular for non- FSZ . Similarly for FS_m -regular and FS_m -irregular. These seem appropriate choices, but for this paper the author will stick with the existing terminology.

Example 1.6. On the other hand, Iovanov et al. [3] also established that non- FSZ groups exist by using GAP [2] to show that there are exactly 32 groups of order 5^6 which are not FSZ_5 . We will also make use of these GAP functions in the subsequent.

Example 1.7. The author has constructed examples of non- FSZ_{p^j} p -groups for all primes $p > 3$ and $j \in \mathbb{N}$ in [5]. For $j = 1$ these groups have order p^{p+1} , which is the minimum possible order for any non- FSZ p -group. Combined, [3, 5, 12] show, among other things, that the minimum order of non- FSZ 2-groups is at least 2^{10} ,

and the minimum order for non- FSZ 3-groups is at least 3^8 . It is unknown if any non- FSZ 2-groups or 3-groups exist, however.

Example 1.8. Schauenburg [12] provides several equivalent formulations of the FSZ_m properties, and uses them to construct GAP [2] functions which are useful for testing the property. Using these functions, it was shown that the Chevalley group $G_2(5)$ is not FSZ_5 . In a private communication, Schauenburg has also established that the Harada-Norton simple group HN is also not FSZ_5 . These groups were attacked directly, using advanced computing resources for HN . We will later present an alternative way of using GAP to prove these groups, and their Sylow 5-subgroups, are not FSZ_5 in a way that can be executed more easily.

One consequence of these examples is that the smallest known order for a non- FSZ group is $5^6 = 15,625$. The groups with order divisible by p^{p+1} for $p > 5$ that are readily available through GAP are small in number, problematically large, and do not have convenient representations—matrix groups are usually too memory intensive for what we need to do, so we need permutation or polycyclic presentations for accessible calculations. For this reason, all of the examples we pursue in the following sections will hone in on the non- FSZ_5 property for groups with order divisible by 5^6 . In most of the examples, this is the largest power of 5 dividing the order, with the Monster group and the perfect groups of order $12 \cdot 5^7$ being the exceptions.

2. OBTAINING THE NON- FSZ PROPERTY FROM CERTAIN SUBGROUPS

Our first elementary result offers a starting point for investigating non- FSZ_m groups of minimal order.

Lemma 2.1. *Let G be a group with minimal order in the class of non- FSZ_m groups. Then $|G_m(u, g)| \neq |G_m(u, g^n)|$ for some $(n, |G|) = 1$ implies $g \in Z(G)$.*

Proof. If not then $C_G(g)$ is a smaller non- FSZ_m group, a contradiction. \square

The result applies to non- FSZ_m groups in a class that is suitably closed under the taking of centralizers. For example, we have the following version for p -groups.

Corollary 2.2. *Let P be a p -group with minimal order in the class of non- FSZ_{p^j} p -groups. Then $|P_{p^j}(u, g)| \neq |P_{p^j}(u, g^n)|$ for some $p \nmid m$ implies $g \in Z(P)$.*

Example 2.3. From the examples in the previous section, we know the minimum possible order for a non- FSZ_p p -group for $p > 3$ is p^{p+1} . It remains unknown if the examples of non- FSZ_{p^j} p -groups from [5] for $j > 1$ have minimal order among non- FSZ_{p^j} p -groups.

Next, we determine a condition for when the non- FSZ property for a normal subgroup implies the non- FSZ property for the full group.

Lemma 2.4. *Let G be a group and suppose H is a non- FSZ_m normal subgroup with m coprime to $[G : H]$. Then G is non- FSZ_m .*

Proof. Let $u, g \in H$ and $(n, |g|) = 1$ be such that $|H_m(u, g)| \neq |H_m(u, g^n)|$. By the index assumption, for all $x \in G$ we have $x^m \in H \Leftrightarrow x \in H$, so by definitions $G_m(u, g) = H_m(u, g)$ and $G_m(u, g^n) = H_m(u, g^n)$, which gives the desired result. \square

Corollary 2.5. *Let G be a finite group and suppose P is a normal non- FSZ_{p^j} Sylow p -subgroup of G for some prime p . Then G is non- FSZ_{p^j} .*

Corollary 2.6. *Let G be a finite group and P a non- FSZ_{p^j} Sylow p -subgroup of G . Then the normalizer $N_G(P)$ is non- FSZ_{p^j} .*

The following is the conjectural relation we propose for the FSZ property. It is a generalization of the preceding corollary. The remaining sections of this paper are dedicated to constructing additional examples in support of it.

Conjecture 2.7. A group is FSZ if and only if all of its Sylow subgroups are FSZ .

Some remarks on why this conjecture may involve some deep results to establish affirmatively seems in order. Consider a group G and let $u, g \in G$ and $n \in \mathbb{N}$ with $(n, |G|) = 1$. Suppose that g has order a power of p , for some prime p . Then

$$G_{p^j}(u, g) = \bigcup G_{p^j}(u, g) \cap P^x,$$

where the union runs over all distinct conjugates P^x in $C_G(g)$ of a fixed Sylow p -subgroup P of $C_G(g)$. Let $P_{p^j}^x(u, g) = G_{p^j}(u, g) \cap P^x$. Then $|G_{p^j}(u, g)| = |P_{p^j}^x(u, g^n)|$ if and only if there is a bijection $\bigcup P^x(u, g) \rightarrow \bigcup P^x(u, g^n)$. In the special case $u \in P$, if P was FSZ_{p^j} we would have a bijection $P(u, g) \rightarrow P(u, g^n)$, but this does not obviously guarantee a bijection $P^x(u, g) \rightarrow P^x(u, g^n)$ for all conjugates. Attempting to get a bijection $\bigcup P^x(u, g) \rightarrow \bigcup P^x(u, g^n)$ amounts, via the Inclusion-Exclusion Principle, to controlling the intersections of any number of conjugates and how many elements those intersections contribute to $G_{p^j}(u, g)$ and $G_{p^j}(u, g^n)$. There is no easy or known way to predict the intersections of a collection of Sylow p -subgroups for a completely arbitrary G , so any positive affirmation of the conjecture will impose a certain constraint on these intersections. Moreover, we have not considered the case of the sets $G_m(u, g)$ where m has more than one prime divisor, nor those cases where u, g do not have prime power order, so a positive affirmation of the conjecture is also expected to show that the FSZ_m properties are all derived from the FSZ_{p^j} properties for all prime powers dividing m . On the other hand, a counterexample seems likely to involve constructing a large group which exhibits a complex pattern of intersections in its Sylow p -subgroups for some prime p , or otherwise exhibits the first example of a group which is FSZ_{p^j} for all prime powers but is nevertheless not FSZ .

Example 2.8. All currently known non- FSZ groups are either p -groups (for which the conjecture is trivial), or come from perfect groups (though the relevant centralizers need not be perfect). The examples of both FSZ and non- FSZ groups we establish here will also all come from perfect groups and p -groups.

3. THE NON- FSZ SPORADIC SIMPLE GROUPS

The goal for this section is to show that the Chevalley group $G_2(5)$, and all sporadic simple groups with order divisible by 5^6 , as well as their Sylow 5-subgroups, are non- FSZ_5 . We begin with a discussion of the general idea for the approach.

Our first point of observation is that the only primes p such that p^{p+1} divides the order of any of these groups have $p \leq 5$. Indeed, a careful analysis of the non- FSZ groups of order 5^6 found in [3] shows that several of them are non-split extensions with a normal extra-special group of order 5^5 , which can be denoted in AtlasRep notation as $5^{1+4}.5$. Consulting the known maximal subgroups for these groups we

can easily infer that the Sylow 5-subgroups of HN , $G_2(5)$, B , and Ly have this same form, and that the Monster has such a p -subgroup. Indeed, the Monster's Sylow 5-subgroup has the form $5^{1+6}.5^2$, a non-split extension of the elementary abelian group of order 25 by an extra special group of order 5^7 . Given this, we suspect that these Sylow 5-subgroups are all non- FSZ_5 , and that this will cause the groups themselves to be non- FSZ_5 .

We can then exploit the fact that non-trivial p -groups all have non-trivial centers to obtain centralizers in the parent group that contain a Sylow 5-subgroup. In the case of $G = HN$ or $G = G_2(5)$, we can find $u, g \in P$, a Sylow 5-subgroup of G , with $|P_5(u, g)| \neq |P_5(u, g^2)|$, and show that for $H = C_G(g)$ we have $|H_5(u, g)| \neq |H_5(u, g^2)|$. Since necessarily $|H_5(u, g)| = |G_5(u, g)|$ and $|H_5(u, g^2)| = |G_5(u, g^2)|$, this will show that HN and $G_2(5)$ are non- FSZ_5 . Unfortunately, it turns out that P is not normal in H in either case, so the cardinalities of these sets in H must be checked directly, rather than using the much smaller group P alone.

In the case of the Monster, there is a unique (conjugacy class of a) centralizer with order divisible by 5^9 , so we are free to pick any subgroup G of M that contains a centralizer with order divisible by 5^9 . Fortunately, not only is such a (maximal) subgroup known, but Bray and Wilson [1] have also computed a permutation representation for it. This is available in GAP via the AtlasRep package. This makes all necessary calculations for the Monster accessible. The Sylow 5-subgroup is fairly easily shown to be non- FSZ_5 directly from this. However, the centralizer we get in this way has a very large order, and its Sylow 5-subgroup is not normal, making it impractical to work with on a personal computer. However, further consultation of character tables shows that the Monster group has a unique conjugacy class of an element of order 10 whose centralizer is divisible by 5^6 . So we may again pick any convenient (maximal) subgroup with such a centralizer, and it turns out the same maximal subgroup works. We construct the appropriate element of order 10 by using suitable elements from Sylow subgroups of the larger centralizer, and similarly to get the element u . Again it turns out that the Sylow 5-subgroup of this smaller subgroup is not normal, so we must compute the set cardinalities over the entire centralizer in question. However, this centralizer is about 1/8000-th the size of the initial one, and the cardinalities in question are not particularly unreasonable to compute over a permutation group of this size and degree.

The Baby Monster can then be handled by using the fact that the Monster contains the double cover of B as the centralizer of an involution. The author thanks Robert Wilson for pointing this out. For the Lyons group, the idea is much the same as for HN and $G_2(5)$, with the additional complication that the AtlasRep package does not currently contain any permutation representations. To resolve this, we obtain a permutation representation for Ly , either computed directly in GAP or downloaded, to ultimately construct a suitable permutation representation of the maximal subgroup in question. Once this is done the calculations proceed without difficulties.

While we will make use of the GAP functions from [3, 12], we will at times encounter situations where none of these functions is particularly reasonable to use without requiring large amounts of memory. In the case of the **FSInd** function from [3], the issue is that the function must construct the entire group as a list multiple times, which can easily consume available memory depending on the size of the group and how it is stored. In the case of **FSZtest** from [12], the issue arises

when computing character tables for centralizers, which can also quickly consume available memory. Thus we desire a function which can be used to test the FSZ_m property which is more memory friendly.

We will define a function **FSZSetCards** below, which can be thought of as a variation on **FSInd**, which accepts the five arguments C, u, g, m, n . The intended use case has $1 \neq g \in G$, $C = C_G(g)$, $u \in C$, n coprime to the order of g , and m is the value for which we are testing the FSZ_m property. We will always use the function in a way which guarantees these conditions, so for simplicity no checks are performed to ensure these assumptions are satisfied. The return value is the two element list $[|C_m(u, g)|, |C_m(u, g^n)|]$. Thus, in the intended use case, G will be non- FSZ_m if this list contains two distinct entries.

```

FSZSetCards := function(C,u,g,m,n)
    local contribs, apow, aupow, a;

    contribs := [0,0];

    for a in C do
        apow := a^m;
        aupow := (a*Inverse(u))^m;

        if (apow = g and aupow = g) then
            contribs[1] := contribs[1] + 1;

            elif (apow=g^n and aupow=g^n) then
                contribs[2] := contribs[2] + 1;
            fi;
        od;

    return(contribs);
end;

```

We will also modify the **FSInd** function from [3] slightly so that, instead of returning **false** when the test shows the group is non- FSZ , the function returns the data that established the failure of the FSZ property. In particular, this return value is a three element list $[u, g, n]$ such that $|G_m(u, g)| \neq |G_m(u, g^n)|$ and n is coprime to $|G|$. The values of m, G are provided to **FSInd** as inputs. The return value of **FSZtest** can also be modified in a similar fashion, though we will have no particular need for this modification. Some additional helper functions can be found in the Appendix.

3.1. Chevalley group $G_2(5)$. We now show that $G_2(5)$ and its Sylow 5-subgroups are not FSZ_5 .

Since $G_2(5)$ is of relatively small order, it can be attacked quickly and easily.

Theorem 3.1. *The simple Chevalley group $G_2(5)$ and its Sylow 5-subgroup are non- FSZ_5 .*

Proof. The claims follow from running the following GAP code.

```

G := AtlasGroup("G2(5)");
P := SylowSubgroup(G,5);

# The following shows P is not FSZ_5
ex := FSInd(P,5,5);

u := ex[1];
g := ex[2];
n := ex[3];

C := Centralizer(G,g);

#Check the cardinalities
FSZSetCards(C,u,g,5,n);

```

The output is $[0,625]$, so it follows that G and P are both non- FSZ_5 as desired. \square

We note that P is not normal in C , and indeed C is a perfect group of order $375,000 = 2^3 \cdot 3 \cdot 5^6$.

The call to **FSZSetCards** above runs in approximately 11 seconds, which is approximately the amount of time necessary to run **FSZtest** on $G_2(5)$ directly. In this case, this call is not the most time efficient since the groups in question are of reasonably small sizes and permutation degree.

3.2. The Harada-Norton group. For the group HN the idea proceeds similarly.

Theorem 3.2. *The Harada-Norton simple group HN and its Sylow 5-subgroup are not FSZ_5 .*

Proof. To establish the claims it suffices to run the following GAP code.

```

G := AtlasGroup("HN");
P := SylowSubgroup(G,5);

# G, thus P, has very large degree.
# Polycyclic groups are easier to work with.
hp := IsomorphismPcGroup(P);
P := Image(hp);

ex := FSInd(P,5,5);

u := Image(InverseGeneralMapping(hp),ex[1]);
g := Image(InverseGeneralMapping(hp),ex[2]);
n := ex[3];

C := Centralizer(G,g);
hc := IsomorphismPcGroup(C);
C := Image(hc);

```



```
FSZSetCards(C, Image(hc, u), Image(hc, g), 5, n);
```

This code executes in approximately 15 minutes, with around 12 of them spent finding P . The final output is $[0, 3125]$, so we find that both P and HN are non- FSZ_5 , as desired. \square

P is again not a normal subgroup of C , so we again must test the entire centralizer rather than just P . We note that $|C| = 2^5 5^6 = 500,000$. Indeed, C is itself non- FSZ_5 of necessity, and the fact that the call to `IsomorphismPcGroup` did not fail means that C is solvable, and in particular not perfect and not a p -group.

Remark 3.3. Whenever available, a polycyclic presentation is to be preferred over a permutation representation. If we apply `FSZSetCards` before converting C to a polycyclic representation, then the run time goes up substantially.

3.3. The Monster group. We will now consider the Monster group M . The full Monster group is famously difficult to compute in. But, as detailed in the beginning of the section, by consulting character tables of M and its known maximal subgroups, we can find a maximal subgroup which contains a suitable centralizer (indeed, two suitable centralizers) and also admits a known permutation representation [1].

Theorem 3.4. *The Monster group M and its Sylow 5-subgroup are not FSZ_5 .*

Proof. The Sylow 5-subgroup of M has order 5^9 . Consulting the character table of M , we see that M has a unique conjugacy class yielding a centralizer with order divisible by 5^9 , and a unique conjugacy class of an element of order 10 whose centralizer has order divisible by 5^6 ; moreover, the order of the latter centralizer is not divisible by 5^7 . It suffices to consider any maximal subgroups containing such centralizers. The maximal subgroup of shape $5_+^{1+6} : 2.J_2.4$, which is the normalizer associated to a $5B$ class, is one such choice.

We first show that the Sylow 5-subgroup of M is not FSZ_5 .

```
G := AtlasGroup("5^(1+6):2.J2.4");;
P := SylowSubgroup(G, 5);;
hp := IsomorphismPcGroup(P);;
FSInd(Image(hp), 5, 5);
```

The function `FSInd` returns a list of three entries here, establishing that P is not FSZ_5 . The proper centralizer with order divisible by 5^9 is still impractical to work with. So we will use the data for P to construct the element of order 10 mentioned above. We note that $Z(P)$ is cyclic.

```
zp := Center(P).1;;
C := Centralizer(G, zp);;
Q := SylowSubgroup(C, 2);;

zq := Filtered(Center(Q), q -> Order(q) > 1 and
Size(Centralizer(G, zp*q)) = 12000000)[1];;
```

```
#Q's center is a Klein 4-group.
#zq should be one of Center(Q).1 or Center(Q).2, though
#we can't guarantee it will always be the same
```

```

#one on every running of the code.

#This gives us the g and centralizer we want.
g := zp*zq;;
C := Centralizer(G,g);;

#Reducing the permutation degree will
#save a lot of computation time later.
isoc := SmallerDegreePermutationRepresentation(C);;
C := Image(isoc);;
g := Image(isoc,g);;
zq := Image(isoc,zq);;

#Now proceed to construct a choice of u.
P:=SylowSubgroup(C,5);;
hp:=IsomorphismPcGroup(P);;
P:=Image(hp);;

# The (smaller) Sylow 5-subgroup is still not FSZ-5.
ex := FSInd(P,5,5);

up:=Image(InverseGeneralMapping(hp),ex[1]);;

#Define our choice of u.
#In this case, u has order 50.
u:=up*zq;;

#Finally, we compute the cardinalities
# of the relevant sets.
FSZSetCards(C,u,g,5,7);

```

This final function yields $[0,15000]$, which proves that M is not FSZ_5 , as desired. \square

This final function call takes approximately 45 minutes to complete. The conversion of C to a lower degree takes about 27 minutes, yielding a representation of degree 18,125 in the author's case. However, it requires approximately 2.5 GB to complete. This conversion can be skipped to keep the memory demands well under 2GB, but the execution time for `FSZSetCards` will inflate to approximately a day and a half.

Remark 3.5. In the first definition of C above, containing the full Sylow 5-subgroup of M , we have $|C| = 9.45 \times 10^{10} = 2^8 \cdot 3^3 \cdot 5^9 \cdot 7$. For the second definition of C , corresponding to the centralizer of an element of order 10, we have $|C| = 1.2 \times 10^7 = 2^8 \cdot 3 \cdot 5^6$. The first centralizer is thus $7875 = 3^2 \cdot 5^3 \cdot 7$ times larger than the second one. Either one is many orders of magnitude smaller than $|M| \approx 8.1 \times 10^{53}$, but the larger one was still too large to work with for practical purposes.

3.4. The Baby Monster. We can now consider the Baby Monster B .

Theorem 3.6. *The Baby Monster B and its Sylow 5-subgroup are both non- FSZ_5 .*

Proof. The Baby Monster is well known to have a maximal subgroup of the form $HN.2$, so it follows that B and HN have isomorphic Sylow 5-subgroups. Since we have previously established that HN has a non- FSZ_5 Sylow 5-subgroup, this immediately gives the claim about the Sylow 5-subgroup of B .

From the character table of B we see that there is a unique non-identity centralizer with order divisible by 5^6 . This corresponds to an element of order 5 from the 5B class, and the centralizer has order $6,000,000 = 2^7 \cdot 3 \cdot 5^6$. In the double cover $2.B$ of B , this centralizer is covered by the centralizer of an element of order 10; this centralizer necessarily has order $12,000,000$. Since M contains $2.B$ as a maximal subgroup, and there is a unique centralizer of an element of order 10 in M with order divisible by $12,000,000$, these centralizers in $2.B$ and M are isomorphic. We have already computed this centralizer in M in Theorem 3.4. To obtain the centralizer in B , we need only quotient by the central involution. In the notation of the proof of Theorem 3.4, this involution is precisely zq .

GAP will automatically convert this quotient group D into a lower degree representation, yielding a permutation representation of degree 3125 for the centralizer. This will require as much as 8GB of memory to complete. Moreover, the image of zp from Theorem 3.4 in this quotient group yields the representative of the 5B class, denoted here by g . Using the image of up in the quotient for u , we can then easily run `FSZSetCards(C,u,g,5,2)` to get a result of $[15000,3125]$, which shows that B is non- FSZ_5 as desired. This final call completes in about 4 minutes. \square

Remark 3.7. Note that in M the final return values summed to 15,000, with one of the values 0, whereas in B they sum to 18,125 and neither is zero. This reflects how there is no clear relationship between the *FSZ* properties of a group and its quotients, even when the quotient is by a (cyclic) central subgroup. In particular, it does not immediately follow that the quotient centralizer would yield the non-*FSZ* property simply because the centralizer in M did, or vice versa.

3.5. The Lyons group. There is exactly one other sporadic group with order divisible by 5^6 (or p^{p+1} for $p > 3$): the Lyons group Ly . It is well-known that Ly contains a copy of $G_2(5)$ as a maximal subgroup, and that the order of Ly is not divisible by 5^7 . Therefore Ly and $G_2(5)$ have isomorphic Sylow 5-subgroups, and we have previously established that this Sylow subgroup is not FSZ_5 .

Theorem 3.8. *The maximal subgroup of Ly of the form $5^{1+4} : 4.S_6$ has a faithful permutation representation on 3,125 points, given by the action on the cosets of $4.S_6$. Moreover, this maximal subgroup, Ly , and their Sylow 5-subgroups are all non- FSZ_5 .*

Proof. Checking the character table for Ly as before, we find there is a unique non-identity conjugacy class whose corresponding centralizer has order divisible by 5^6 . In particular, the order of this centralizer is $2,250,000 = 2^4 \cdot 3^2 \cdot 5^6$, and it comes from an element of order 5. So any maximal subgroup containing a centralizer of an element of order 5 with this group order will suffice. The maximal subgroup $5^{1+4} : 4.S_6$ is the unique such choice.

The new difficulty here is that, by default, there are only matrix group representations available though the AtlasRep package for Ly and $5^{1+4} : 4.S_6$, which

are ill-suited for our purposes. However, faithful permutation representations for Ly are known, and they can be constructed through GAP with sufficient memory available provided one uses a well-chosen method. A detailed description of how to acquire the permutation representation on 8,835,156 points, as well as downloads for the generators (including MeatAxe versions courtesy of Thomas Breuer) can be found on the web, courtesy Pfeiffer [11].

Using this, we can then obtain a permutation representation for the maximal subgroup $5^{1+4} : 4.S_6$ on 8,835,156 points using the programs available on the online ATLAS [14]. This in turn is fairly easily converted into a permutation representation on a much smaller number of points, provided one has up to 8 GB of memory available, via `SmallerDegreePermutationRepresentation`. The author obtained a permutation representation on 3125 points, corresponding to the action on the cosets of $4.S_6$. The exact description of the generators is fairly long, so we will not reproduce them here. The author is happy to provide them upon request. One can also proceed in a fashion similar to some of the cases handled in [1] to find such a permutation representation.

Once this smaller degree representation is obtained, it is then easy to apply the same methods as before to show the desired claims about the FSZ_5 properties. We can directly compute the Sylow 5-subgroup, then find u, g through `FSInd`, set C to be the centralizer of g , then run `FSZSetCards(C,u,g,5,2)`. This returns [5000,625], which gives the desired non- FSZ_5 claims. \square

Indeed, `FSZtest` can be applied to (both) the centralizer and the maximal subgroup once this permutation representation is obtained. This will complete quickly, thanks to the relatively low orders and degrees involved.

We also note that the centralizer C so obtained will not have a normal Sylow 5-subgroup, and is a perfect group. The maximal subgroup in question is neither perfect nor solvable, and does not have a normal Sylow 5-subgroup.

4. THE FSZ SPORADIC SIMPLE GROUPS

We can now show that all other sporadic simple groups and their Sylow subgroups are FSZ .

Example 4.1. Any group which is necessarily FSZ (indeed, FSZ^+) by [3, Corollary 5.3] necessarily has all of its Sylow subgroups FSZ , and so satisfies the conjecture. This implies that all of the following sporadic groups, as well as their Sylow p -subgroups, are FSZ (indeed, FSZ^+).

- The Mathieu groups $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$.
- The Janko groups J_1, J_2, J_3, J_4 .
- The Higman-Simms group HS .
- The McLaughlin group McL .
- The Held group He .
- The Rudvalis group Ru .
- The Suzuki group Suz .
- The O’Nan group ON .
- The Conway group Co_3 .
- The Thompson group Th .
- The Tits group ${}^2F_4(2)'$.

Example 4.2. Continuing the last example, it follows that the following are the only sporadic simple groups not immediately in compliance with the conjecture thanks to [3, Corollary 5.3].

- The Conway groups Co_1, Co_2 .
- The Fischer groups $Fi_{22}, Fi_{23}, Fi'_{24}$.
- The Monster M .
- The Baby Monster B .
- The Lyons group Ly .
- The Harada-Norton group HN .

The previous section showed that the last four groups were all non- FSZ_5 and have non- FSZ_5 Sylow 5-subgroups, and so conform to the conjecture. By exponent considerations the Sylow subgroups of the Conway and Fischer groups are all FSZ^+ . The function `FSZtest` can be used to quickly show that Co_1, Co_2, Fi_{22} , and Fi_{23} are FSZ , and so conform to the conjecture.

This leaves just the largest Fischer group Fi'_{24} .

Theorem 4.3. *The sporadic simple group Fi'_{24} and its Sylow subgroups are all FSZ .*

Proof. The exponent of Fi'_{24} can be calculated from its character table and shown to be

$$24516732240 = 2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29.$$

As previously remarked, this automatically implies that the Sylow subgroups are all FSZ (indeed, FSZ^+). By [3, Corollary 5.3] it suffices to show that every centralizer of an element with order not in $\{1, 2, 3, 4, 6\}$ in Fi'_{24} that contains an element of order 16 is FSZ . There is a unique conjugacy class in Fi'_{24} for an element with order (divisible by) 16. The centralizer of such an element has order 32, and is isomorphic to $\mathbb{Z}_{16} \times \mathbb{Z}_2$. So it suffices to consider the elements of order 8 in this centralizer, and show that their centralizers (in Fi'_{24}) are FSZ . Every such element has centralizer of order $1536 = 2^9 \cdot 3$. The exponent is then necessarily $48 = 2^4 \cdot 3$, so by [3, Corollary 5.3] we conclude that these centralizers are FSZ , and so that Fi'_{24} is FSZ as desired.

The following is GAP code verifying these claims.

```
G := AtlasGroup("Fi24'");
GT := CharacterTable("Fi24'");

Positions(OrdersClassRepresentatives(GT) mod 16, 0);

exp := Lcm(OrdersClassRepresentatives(GT));
Collected(FactorsInt(exp));
SetExponent(G, exp);

P := SylowSubgroup(G, 2);

#There are many ways to get an element of order 16.
#Here's a very simple one.
x := Random(P);
```

```

while not Order(x) = 16 do x:=Random(P); od;

C := Centralizer(G,x);;

cents := Filtered(C,y->Order(y)=8);;
cents := List(cents,y->Centralizer(G,y));;

List(cents, Size);

```

□

The following then summarizes our results on sporadic simple groups.

Theorem 4.4. *The following are equivalent for a sporadic simple group G .*

- (1) G is not FSZ .
- (2) G is not FSZ_5 .
- (3) The order of G is divisible by 5^6 .
- (4) G has a non- FSZ Sylow subgroup.
- (5) The Sylow 5-subgroup of G is not FSZ_5 .

Proof. Apply the results of this section and the previous one. □

We have also seen that $G_2(5)$ is a non- FSZ , non-sporadic simple group. It is not known if there are other non- FSZ simple groups.

5. PERFECT GROUPS OF ORDER LESS THAN 10^6

We now look for examples of additional non- FSZ perfect groups. The library of perfect groups stored by GAP has most perfect groups of order less than 10^6 , with a few exceptions noted in the documentation. So we can iterate through the available groups, of which there are 1097. We can use the function `IMMtests` from the appendix to show that most of them are FSZ .

```

#Get all available sizes
Glist := Filtered( SizesPerfectGroups(),
                  n->NrPerfectLibraryGroups(n)>0);;

#Get all available perfect groups
Glist := List( Glist,
              n->List([1..NrPerfectLibraryGroups(n)],
                    k->PerfectGroup(IsPermGroup, n, k)) );;

Glist := Flat(Glist);;

#Remove the obviously FSZ ones
Flist := Filtered( Glist, G->not IMMtests(G)=true );;

```

This gives a list of 63 perfect groups which are not immediately dismissed as being FSZ .

Theorem 5.1. *Of the 1097 perfect groups of order less than 10^6 available through the GAP perfect groups library, exactly 7 of them are not *FSZ*, all of which are extensions of A_5 . All seven of them are non- FSZ_5 . Four of them have order $375,000 = 2^3 \cdot 3 \cdot 5^6$, and three of them have order $937,500 = 2^2 \cdot 3 \cdot 5^7$. Their perfect group ids in the library are:*

$$\begin{array}{llll} [375000, 2], & [375000, 8], & [375000, 9], & [375000, 11], \\ [937500, 3], & [937500, 4], & [937500, 5] & \end{array}$$

Proof. Continuing the preceding discussion, we can apply **FSZtest** to the 63 groups in **Flist** to obtain the desired result. This calculation takes approximately two days of total calculation time on the author's computer, but can be easily split across multiple GAP instances. Most of the time is spent on the *FSZ* groups of orders 375,000 and 937,500. \square

On the other hand, we can also consider the Sylow subgroups of all 1097 available perfect groups, and test them for the *FSZ* property.

Theorem 5.2. *If G is one of the 1097 perfect groups of order less than 10^6 available through the GAP perfect groups library, then the following are equivalent.*

- (1) G is not *FSZ*.
- (2) G has a non-*FSZ* Sylow subgroup.
- (3) G has a non- FSZ_5 Sylow 5-subgroup.
- (4) G is not FSZ_5 .

Proof. Most of the GAP calculations we need to perform now are quick, and the problem is easily broken up into pieces, should it prove difficult to compute everything at once. The most memory intensive case requires about 1.7 GB to test. With significantly more memory available than this, the cases can simply be tested by **FSZtest** en masse, which will establish the result relatively quickly—a matter of hours instead of days. We sketch the details here and leave it to the interested reader to construct the relevant code.

Let **Glist** be constructed in GAP as before. Running over each perfect group, we can easily construct their Sylow subgroups. We can then use **IMMtests** from the appendix to eliminate most cases. There are 256 Sylow subgroups, each from a distinct perfect group, for which **IMMtests** is inconclusive; and there are exactly 4 cases where **IMMtests** definitively shows the non-*FSZ* property, which are precisely the Sylow 5-subgroups of each of the non-*FSZ* perfect groups of order 375,000. These 4 Sylow subgroups are all non- FSZ_5 . We can also apply **FSZtestZ** to the Sylow 5-subgroups of the non-*FSZ* perfect groups of order 937,500 to conclude that they are all non- FSZ_5 . All other Sylow subgroups remaining that come from a perfect group of order less than 937,500 can be shown to be *FSZ* by applying **FSZtest** without difficulty. Of the three remaining Sylow subgroups, one has a direct factor of \mathbb{Z}_5 , and the other factor is easily tested and shown to be *FSZ*, whence this Sylow subgroup is *FSZ*. This leaves two other cases, which are the Sylow 5-subgroups of the perfect groups with ids [937500,7] and [937500,8]. The second of these is easily shown to be *FSZ* by **FSZtest**. The first can also be tested by **FSZtest**, but this is the case that requires the most memory and time—approximately 15 minutes and the indicated 1.7 GB. In this case as well the Sylow subgroups are *FSZ*. This completes the proof. \square

APPENDIX A. ADDITIONAL TESTING FUNCTIONS

When checking many groups for the FSZ property it tends to be handy to run the quick, inexpensive tests first, rather than immediately apply `FSZtest`. In this appendix we define two such testing functions.

The first, `FSZtestZ` is identical to `FSZtest`—and uses several of the helper functions found in [12]—except that instead of calculating and iterating over all rational classes it iterates only over the center. This is primarily useful for testing groups with minimal order in a class closed under centralizers, such as in Lemma 2.1 and Corollary 2.2, or any group with non-trivial center that is suspected of failing the FSZ property at a central value. The function returns `false` if it determines G to be non- FSZ , and returns `fail` otherwise since the test is inconclusive by default (though in the case of minimal order examples it is conclusive).

```

FSZtestZ := function(G)
local CT, z , cl , div , d , chi , m, b ;

cl := RationalClasses(Center(G));
cl := List(cl, Representative);
cl := Filtered(cl, c->not Order(c) in [1,2,3,4,6]);

for z in cl do
  div := Filtered(DivisorsInt( Exponent(G)/Order(z) ),
    m->not Gcd(m, Order(z)) in [1,2,3,4,6]);
  if Length(div) < 1 then continue; fi;

  CT := OrdinaryCharacterTable(G);

  for chi in Irr(CT) do
    for m in div do
      if not IsRat( beta(CT, z ,m, chi) )
        then return false;
      fi ;
    od;
  od;
od;

#the test is inconclusive in general
return fail ;

end ;

```

The second function, `IMMtests`, implements most of the more easily checked conditions found in [3] that guarantee the FSZ property, and calls `FSZtestZ` when it encounters a suitable p -group. The function returns `true` if the test conclusively establishes that the group is FSZ , `false` if it conclusively determines the group is non- FSZ , and `fail` otherwise. Note that whenever this function calls `FSZtestZ`

that test is conclusive by Corollary 2.2, so it must adjust a return value of `fail` to `true`.

```

IMMtests := function(G)
    local sz, b, l, p2, p3, po;

    if IsAbelian(G)
        then return true;
    fi;

    sz := Size(G);

    if sz < 100
        then return true;
    fi;

    if IsPGroup(G) then
        #Regular p-groups are always FSZ.

        l := Collected(FactorsInt(sz))[1];

        if l[1] >= l[2] or Exponent(G) = l[1]
            then return true;
        fi;

        sz := Length(UpperCentralSeries(G));

        if l[1] = 2 then
            if l[2] < 10 or sz < 3
                or Exponent(G) < 64
                then return true;
            elif l[2] = 10 and sz >= 3
                then
                    return FSZtestZ(G) <> false;
            fi;
        elif l[1] = 3 then
            if l[2] < 8 or sz < 4
                or Exponent(G) < 27
                then return true;
            elif l[2] = 8 and sz >= 4
                then
                    return FSZtestZ(G) <> false;
            fi;
        elif sz < l[1] + 1
            then return true;

```

```

    elif sz = l[1]+1 and sz=l[2]
    then
        return FSZtestZ(G)<>false;
    fi;
else
    #check the exponent for non-p-groups
    l := FactorsInt(Exponent(G));
    p2 := Length(Positions(l,2));
    p3 := Length(Positions(l,3));
    po := Filtered(l,x->x>3);

    if ForAll(Collected(po),x->x[2]<2) and
        ( (p2 < 4 and p3 < 4)
        or ( p2 < 6 and p3 < 2) )
        then return true;
    fi;
fi;

#tests were inconclusive
return fail ;
end;

```

REFERENCES

- [1] John N. Bray and Robert A. Wilson. Explicit representations of maximal subgroups of the monster. *Journal of Algebra*, 300(2):834 – 857, 2006. ISSN 0021-8693. doi: <http://dx.doi.org/10.1016/j.jalgebra.2005.12.017>. URL <http://www.sciencedirect.com/science/article/pii/S0021869305007313>.
- [2] GAP. GAP – Groups, Algorithms, and Programming, Version 4.8.4. <http://www.gap-system.org>, Jun 2016.
- [3] M. Iovanov, G. Mason, and S. Montgomery. FSZ-groups and Frobenius-Schur indicators of quantum doubles. *Math. Res. Lett.*, 21(4):1–23, 2014.
- [4] Yevgenia Kashina, Yorck Sommerhäuser, and Yongchang Zhu. On higher Frobenius-Schur indicators. *Mem. Amer. Math. Soc.*, 181(855):viii+65, 2006. ISSN 0065-9266. doi: 10.1090/memo/0855. URL <http://dx.doi.org/10.1090/memo/0855>.
- [5] M. Keilberg. Examples of non-FSZ p-groups for primes greater than three. *ArXiv e-prints*, September 2016. under review.
- [6] Marc Keilberg. Higher indicators for some groups and their doubles. *J. Algebra Appl.*, 11(2):1250030, 38, 2012. ISSN 0219-4988. doi: 10.1142/S0219498811005543. URL <http://dx.doi.org/10.1142/S0219498811005543>.
- [7] Marc Keilberg. Higher Indicators for the Doubles of Some Totally Orthogonal Groups. *Comm. Algebra*, 42(7):2969–2998, 2014. ISSN 0092-7872. doi: 10.1080/00927872.2013.775651. URL <http://dx.doi.org/10.1080/00927872.2013.775651>.

- [8] C. Negron and S.-H. Ng. Gauge invariants from the powers of antipodes. *ArXiv e-prints*, September 2016.
- [9] Siu-Hung Ng and Peter Schauenburg. Higher Frobenius-Schur indicators for pivotal categories. In *Hopf algebras and generalizations*, volume 441 of *Contemp. Math.*, pages 63–90. Amer. Math. Soc., Providence, RI, 2007. doi: 10.1090/conm/441/08500. URL <http://dx.doi.org/10.1090/conm/441/08500>.
- [10] Siu-Hung Ng and Peter Schauenburg. Central invariants and higher indicators for semisimple quasi-Hopf algebras. *Trans. Amer. Math. Soc.*, 360(4): 1839–1860, 2008. ISSN 0002-9947. doi: 10.1090/S0002-9947-07-04276-6. URL <http://dx.doi.org/10.1090/S0002-9947-07-04276-6>.
- [11] Markus J. Pfeiffer. Computing a (faithful) permutation representation of Lyons’ sporadic simple group, 2016. URL <https://www.morphism.de/~markusp/posts/2016-06-20-computing-permutation-representation-1>.
- [12] P. Schauenburg. Higher frobenius-schur indicators for drinfeld doubles of finite groups through characters of centralizers. *ArXiv e-prints*, April 2016.
- [13] R. A. Wilson, R. A. Parker, S. Nickerson, J. N. Bray, and T. Breuer. AtlasRep, a gap interface to the atlas of group representations, Version 1.5.1. <http://www.math.rwth-aachen.de/~Thomas.Breuer/atlasrep>, Mar 2016. Refereed GAP package.
- [14] Robert Wilson, Peter Walsh, Jonathan Tripp, Ibrahim Suleiman, Richard Parker, Simon Norton, Simon Nickerson, Steve Linton, John Bray, and Rachel Abbott. ATLAS of Finite Group Representations - Version 3. URL <http://brauer.maths.qmul.ac.uk/Atlas/v3/>. Accessed: October 3, 2016.
E-mail address: keilberg@usc.edu